Information Technology Institute

Mct/ROB/200 Robotics, Spring Term 12-13

Lecture 4 – Friday March 15, 2012

Inverse Kinematics

These slides are based on materials from the following books:

- Saeed Benjamin Niku. *Introduction to Robotics*. 2nd Ed., Wiley, 2011.
- Mark W. Spong, Seth Hutchinson, and M. Vidyasagar. *Robot Dynamics and Control*. 2004.
- P. Mckerrow. *Introduction to Robotics*. 1st Ed., Addison-Wesley, 1991.

Objectives

When you have finished this lecture you should be able to:

- Learn how to derive the inverse kinematic equations of the robot.
- Understand how to decouple the inverse kinematics problem into two simpler problems, known respectively, as inverse position kinematics, and inverse orientation kinematics.

Outline

- Inverse Kinematics
- Trigonometric Solutions
- Algebraic Solutions
- Kinematic Decoupling
- Summary

Outline

- **<u>Inverse Kinematics</u>**
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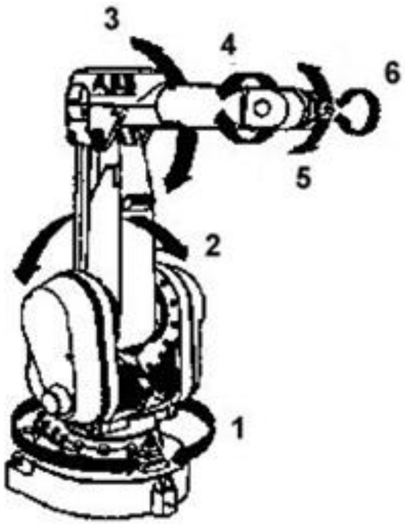
Inverse Kinematics

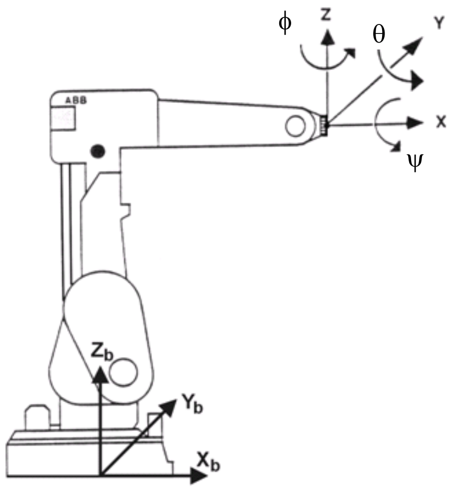


Inverse Kinematics

Given: The position of some point on the robot

Required: The angles of each joint needed to obtain that position

$(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6)$



Given:

x, y, z,ϕ,θ,ψ

Required:

$\theta_k=f_k\ (x,y,z,\phi,\theta,\psi)$

K= 1,…..n (n is DOF = 6)

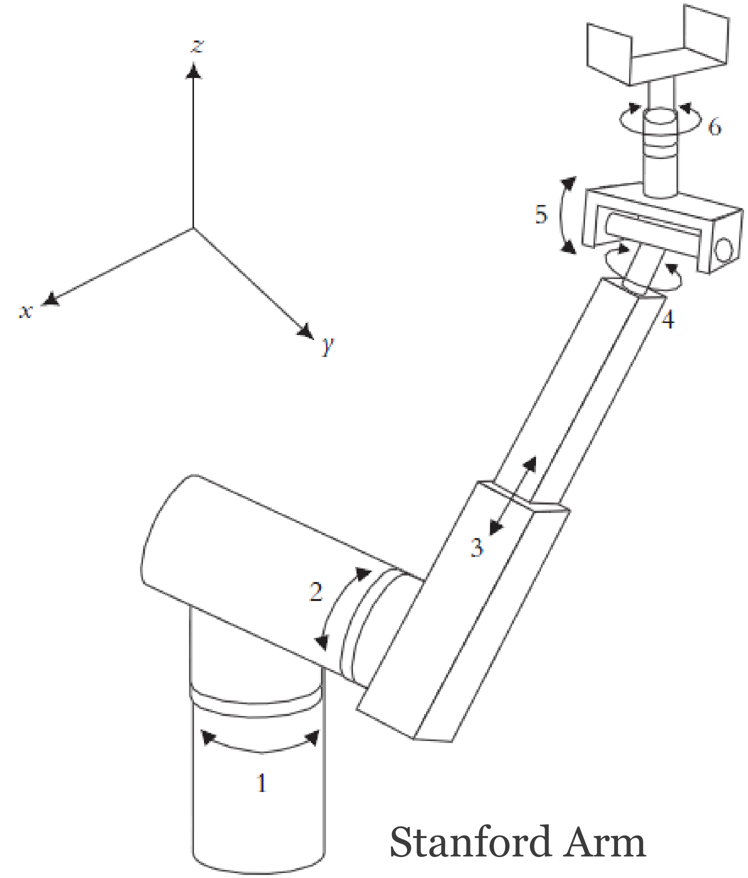
Inverse Kinematics

Example:

Given:

The desired position and orientation of the final frame:

$$
{}^{R}T_{H} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_x & a_x & p_y \\ n_z & o_x & a_x & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



Stanford Arm

Required:

Find the corresponding joint variables θ_1, θ_2, d_3, θ_4, θ_5, and θ_6

Inverse Kinematics

Example (cont'd):

Solution: To find the corresponding joint variables, we must solve the following simultaneous set of **nonlinear trigonometric** equations:

$$n_x = C_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] - S_1(S_4C_5C_6 + C_4S_6)$$

$$n_y = S_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] + C_1(S_4C_5C_6 + C_4S_6)$$

$$n_z = -S_2(C_4C_5C_6 - S_4S_6) - C_2S_5C_6$$

$$o_x = C_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] - S_1(-S_4C_5S_6 + C_4C_6)$$

$$o_y = S_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] + C_1(-S_4C_5S_6 + C_4C_6)$$

$$o_z = S_2(C_4C_5S_6 + S_4C_6) + C_2S_5S_6$$

$$a_x = C_1(C_2C_4S_5 + S_2C_5) - S_1S_4S_5$$

$$a_y = S_1(C_2C_4S_5 + S_2C_5) + C_1S_4S_5$$

$$a_z = -S_2C_4S_5 + C_2C_5$$

$$p_x = C_1S_2d_3 - S_1d_2$$

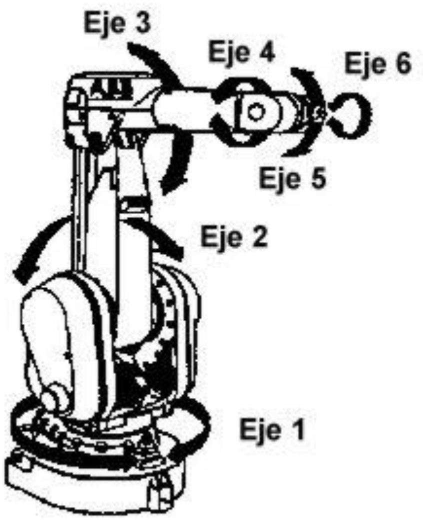
$$p_y = S_1S_2d_3 + C_1d_2$$

$$p_z = C_2d_3$$

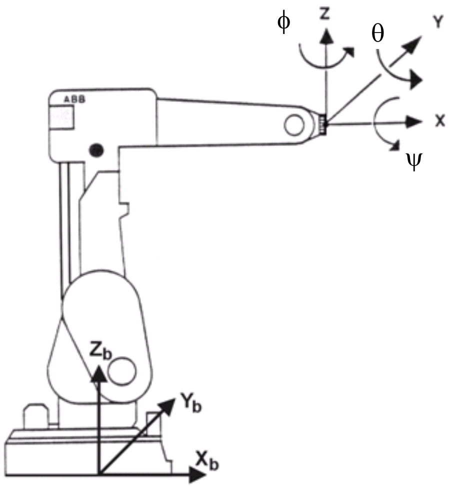
Inverse Kinematics

- The equations in the preceding example are, of course, much **too difficult** to solve directly in closed form. This is the case for most robot arms.
- Therefore, we need to develop efficient and systematic techniques that exploit the particular kinematic structure of the manipulator.
- Whereas the **forward kinematics** problem always has a **unique solution** that can be obtained simply by evaluating the forward equations, the **inverse kinematics** problem **may or may not have a solution**. Even if a solution exists, it **may or may not be unique**.

Inverse Kinematics



Inverse Kinematics



Solutions

Trigonometric

Algebraic

Outline

- Inverse Kinematics
- **Trigonometric Solutions**
- Algebraic Solutions
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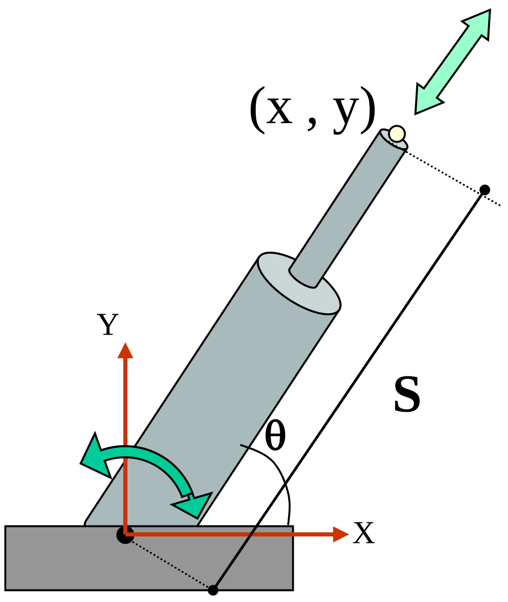
- Summary

Trigonometric Solutions

- **2-DOF Arm**

Given: x, y

Required: θ, S



Solution:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad S = \sqrt{(x^2 + y^2)}$$

See Trigonometric Solution.xls posted on the course website

Trigonometric Solutions

- **2-DOF Revolute Arm**

Given: x, y, l_1, l_2

Required: θ_1, θ_2

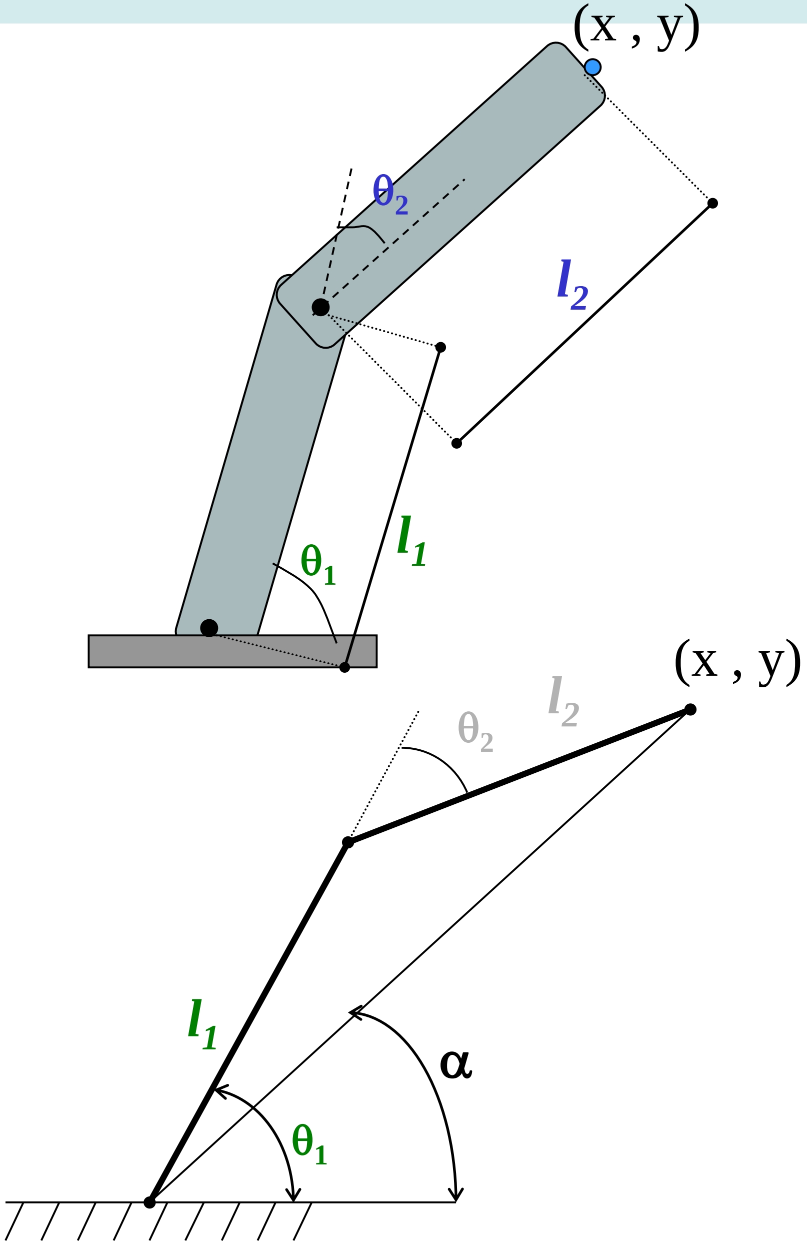
Solution:

$$\theta_2 = \cos^{-1}\left(\frac{x^2 + y^2 - l_1^{\ 2} - l_2^{\ 2}}{2l_1 l_2}\right)$$

Redundant because θ_2 can take positive or negative values

$$\theta_1 = \sin^{-1}\left(\frac{l_2 \sin(\theta_2)}{\sqrt{x^2 + y^2}}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

Redundant because θ_1 has two possible values



Outline

- Inverse Kinematics
- Trigonometric Solutions
- **Algebraic Solutions**
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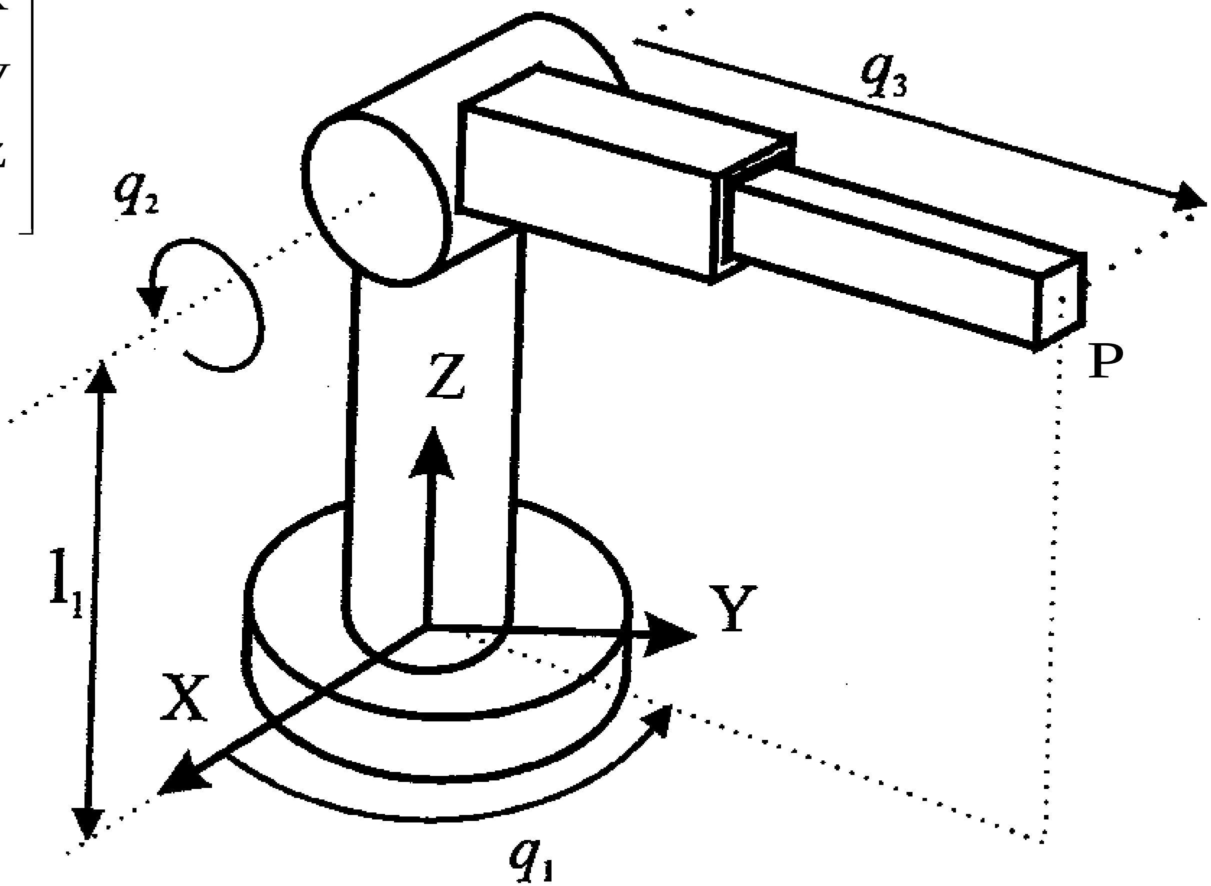
Algebraic Solutions

- **3-DOF Polar Robot**

Given:

$$P = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Required: q_1, q_2, q_3



Algebraic Solutions

- **3-DOF Polar Robot**

Solution:

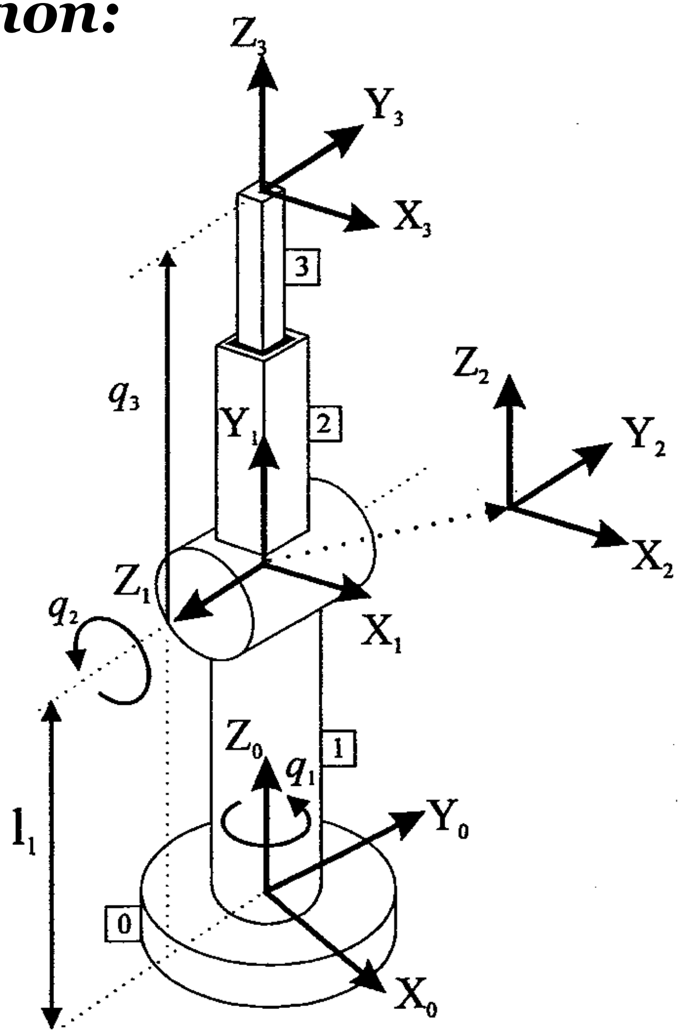


Table of D-H Parameters

	Rz	Tz	Tx	Rx
i	θ_i	d_i	a_i	α_i
1	q_1	l_1	0	90°
2	q_2	0	0	-90°
3	0	q_3	0	0

Algebraic Solutions

- **3-DOF Polar Robot**

$$ {}^{i-1}A_i = \begin{bmatrix} C\theta_i & -C\alpha_i S\theta_i & S\alpha_i S\theta_i & a_i C\theta_i \\ S\theta_i & C\alpha_i C\theta_i & -S\alpha_i C\theta_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

i	θ_i	d_i	a_i	α_i
1	q_1	l_1	0	90º
2	q_2	0	0	-90º
3	0	q_3	0	0

$$ {}^{0}A_1 = \begin{bmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{1}A_2 = \begin{bmatrix} C_2 & 0 & -S_2 & 0 \\ S_2 & 0 & C_2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{2}A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}A_2 = \begin{bmatrix} C_1C_2 & -S_1 & -C_1S_2 & 0 \\ S_1C_2 & C_1 & -S_1S_2 & 0 \\ S_2 & 0 & C_2 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ T = {}^{0}A_3 = \begin{bmatrix} C_1C_2 & -S_1 & -C_1S_2 & -q_3C_1S_2 \\ S_1C_2 & C_1 & -S_1S_2 & -q_3S_1S_2 \\ S_2 & 0 & C_2 & q_3C_2 + l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

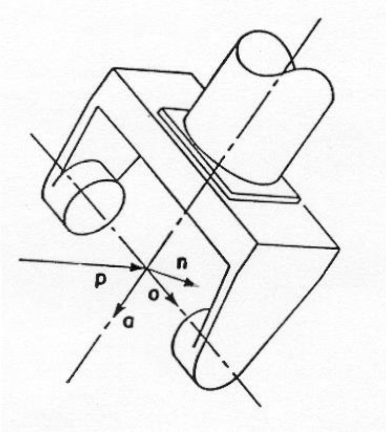
Algebraic Solutions

- **3-DOF Polar Robot**

$$^{0}T_{3} = {^{0}A_{1}} \cdot {^{1}A_{2}} \cdot {^{2}A_{3}} \equiv \Rightarrow {^{0}T_{3}} = \begin{bmatrix} \mathbf{n} & \mathbf{o} & \mathbf{a} & \mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\equiv \Downarrow$$

$$T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



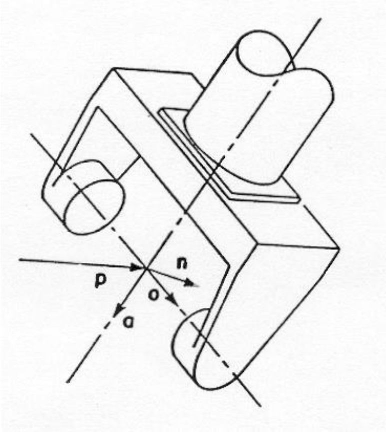
Algebraic Solutions

- **3-DOF Polar Robot**

$$T = A_1 \cdot A_2 \cdot A_3 \Rightarrow T = \begin{bmatrix} \mathbf{n} & \mathbf{o} & \mathbf{a} & \mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Algebraic Solutions

- **3-DOF Polar Robot**

$$T = A_1 \cdot A_2 \cdot A_3 \quad \Longrightarrow \quad A_1^{-1} \cdot T = A_1^{-1} \cdot A_1 \cdot A_2 \cdot A_3$$

Premultiply by A_1^{-1}

$$A_1^{-1} \cdot T = A_2 \cdot A_3$$

$$A_1^{-1} = \begin{bmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} C_1 & S_1 & 0 & 0 \\ 0 & 0 & 1 & -l_1 \\ S_1 & -C_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note: ONLY for homogenous transformation matrices

$$\mathbf{A} = \begin{bmatrix} \mathbf{R} & \mathbf{T} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{R}^T & \mathbf{R}^T \times (-\mathbf{T}) \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Algebraic Solutions

- **3-DOF Polar Robot**

$$A_1^{-1} \cdot \mathrm{T} = A_2 \cdot A_3$$

$$= \begin{bmatrix} C_1 & S_1 & 0 & 0 \\ 0 & 0 & 1 & -l_1 \\ S_1 & -C_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & \mathrm{o_x} & \mathrm{a_x} & \mathrm{p_x} \\ n_y & \mathrm{o_y} & \mathrm{a_y} & \mathrm{p_y} \\ n_z & \mathrm{o_z} & \mathrm{a_z} & \mathrm{p_z} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_2 & 0 & -S_2 & 0 \\ S_2 & 0 & C_2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_2 & 0 & -S_2 & -S_2 q_3 \\ S_2 & 0 & C_2 & C_2 q_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Algebraic Solutions

- **3-DOF Polar Robot**

$$\begin{bmatrix} C_1 & S_1 & 0 & 0 \\ 0 & 0 & 1 & -l_1 \\ S_1 & -C_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_2 & 0 & -S_2 & -S_2 q_3 \\ S_2 & 0 & C_2 & C_2 q_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From these 12 relations, take those express q_1 as function of constants,

which is element (3,4).

$$S_1 p_x - C_1 p_y = 0 \Rightarrow \tan(q_1) = \frac{p_y}{p_x} \Rightarrow q_1 = \tan^{-1}\left(\frac{p_y}{p_x}\right)$$

Algebraic Solutions

- **3-DOF Polar Robot**

$$A_1^{-1} \cdot T = A_2 \cdot A_3$$

Premultiply by A_2^{-1}

$$A_2^{-1} \cdot A_1^{-1} \cdot T = A_2^{-1} \cdot A_2 \cdot A_3$$

$$A_2^{-1} \cdot A_1^{-1} \cdot T = A_3$$

$$A_2^{-1} = \begin{bmatrix} C_2 & 0 & -S_2 & 0 \\ S_2 & 0 & C_2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} C_2 & S_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -S_2 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Algebraic Solutions

- **3-DOF Polar Robot**

$$A_2^{-1} \cdot A_1^{-1} \cdot T = A_3$$

$$= \begin{bmatrix} C_2 & S_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -S_2 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 & S_1 & 0 & 0 \\ 0 & 0 & 1 & -l_1 \\ S_1 & -C_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_2 C_1 & C_2 S_1 & S_2 & -l_1 S_2 \\ -S_1 & C_1 & 0 & 0 \\ -S_2 C_1 & -S_1 S_2 & C_2 & -C_2 l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Algebraic Solutions

- **3-DOF Polar Robot**

$$\begin{bmatrix} C_2C_1 & C_2S_1 & S_2 & -l_1S_2 \\ -S_1 & C_1 & 0 & 0 \\ -S_2C_1 & -S_1S_2 & C_2 & -C_2l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Considering the element (1,4), we arrive at:

$$C_2C_1p_x + C_2S_1p_y + S_2p_z - l_1S_2 = 0$$

$$\Downarrow$$

$$C_2(C_1p_x + S_1p_y) + S_2(p_z - l_1) = 0 \Rightarrow \tan(q_2) = \frac{C_1p_x + S_1p_y}{(l_1 - p_z)}$$

Algebraic Solutions

- **3-DOF Polar Robot**

$$\tan(q_2) = \frac{C_1 p_x + S_1 p_y}{(l_1 - p_z)}$$

Considering that (see page 23): $S_1 p_x - C_1 p_y = 0$

$$(S_1 p_x - C_1 p_y)^2 = S_1{}^2 p_x{}^2 - C_1{}^2 p_y{}^2 - 2S_1 C_1 p_x p_y = 0$$

$$(1 - C_1^2) p_x^2 + (1 - S_1^2) p_y^2 = 2S_1 C_1 p_x p_y$$

$$C_1{}^2 p_x^2 + S_1^2 p_y^2 + 2S_1 C_1 p_x p_y = p_x^2 + p_y^2$$

$$C_1 \; p_x + S_1 \; p_y = \sqrt{p_x^2 + p_y^2}$$

Note:

(sinθ)²=sinθsinθ=[1-(cosθ)²]

(cosθ)²=cosθcosθ=[1-(sinθ)²]

$$q_2 = \tan^{-1} \frac{\sqrt{p_x{}^2 + p_y{}^2}}{(l_1 - p_z)}$$

Algebraic Solutions

- **3-DOF Polar Robot**

$$A_2^{-1} \cdot A_1^{-1} \cdot T = A_3$$

$$\begin{bmatrix} C_2C_1 & C_1S_1 & S_2 & -l_1S_2 \\ -S_1 & C_1 & 0 & 0 \\ -S_2C_1 & -S_1S_2 & C_2 & -C_2l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Taking the element (3,4), we arrive at:

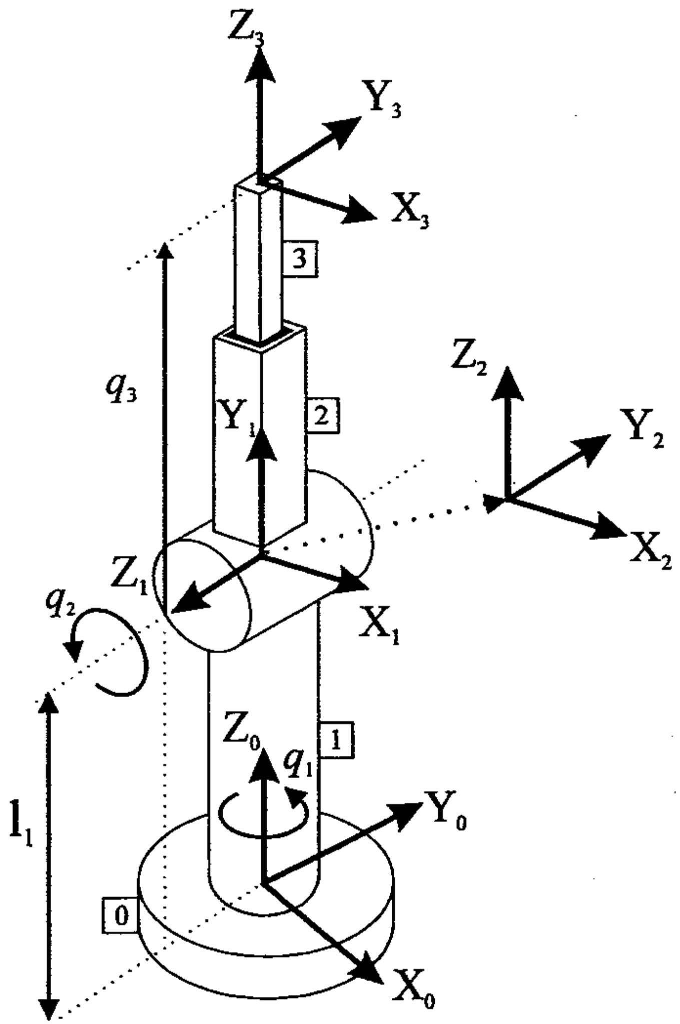
$$-S_2C_1p_x - S_2S_1p_y + C_2p_z - l_1C_2 = q_3$$

$$C_2(p_z - l_1) - S_2(C_1p_x + S_1p_y) = q_3$$

$$q_3 = C_2(p_z - l_1) - S_2\sqrt{p_x^2 + p_y^2}$$

Algebraic Solutions

- **3-DOF Polar Robot**



Inverse Kinematics Solution

$$q_1 = \tan^{-1}\left(\frac{p_y}{p_x}\right)$$

$$q_2 = \tan^{-1}\frac{\sqrt{p_x^{\ 2} + p_y^{\ 2}}}{(l_1 - p_z)}$$

$$q_3 = C_2(p_z - l_1) - S_2\sqrt{p_x^{\ 2} + p_y^{\ 2}}$$

Algebraic Solutions

- **Inverse Kinematic Heuristic**
  1. Equate the total transformation matrix to the manipulator matrix that describes the desired position and orientation of the final frame.
  2. Look at both matrices for:
     a) Elements which contain only one joint variable;
     b) Pairs of elements which will produce an expression in only one joint variable when divided. In particular look for divisions that result in the atan2 function;
     c) Elements, or combinations of elements, that can be simplified using trigonometric identities.

Algebraic Solutions

- **Inverse Kinematic Heuristic**

3. Having selected an element, equate it to the corresponding element in the other matrix to produce an equation. Solve this equation to find a description of one joint variable in terms of the elements of the general transformation matrix.
4. Repeat step 3 until all the elements identified in step 2 have been used.
5. If any of these solutions suffer from inaccuracies, undefined results, or redundant results, set them aside and look for better solutions.

Algebraic Solutions

- **Inverse Kinematic Heuristic**

6. If there are more joint angles to be found, premultiply both sides of the matrix equation by the inverse of the adjacent matrix **A** for the first link to produce a new set of equivalent matrix elements. Alternatively, you can postmultiply both sides by the inverse of the matrix A for the last link in the manipulator, if you think doing so will lead to simpler results.
7. Repeat Steps 2 to 6 until either solutions to all the joint variables have been found, or you have run out of **A** matrices to premultiply (or postmultiply).

Algebraic Solutions

- **Inverse Kinematic Heuristic**

8. If suitable solution cannot be found for a joint variable, choose one of those discarded in step 5, taking note of regions where problems may occur.
9. If a solution cannot be found for a joint variable in terms of the elements of the manipulator transform, it may be that the manipulator cannot achieve the specified position and orientation: the position is outside the manipulator's workspace. Also, theoretical solutions may not be physically attainable because of mechanical limits on the range of joint variable.

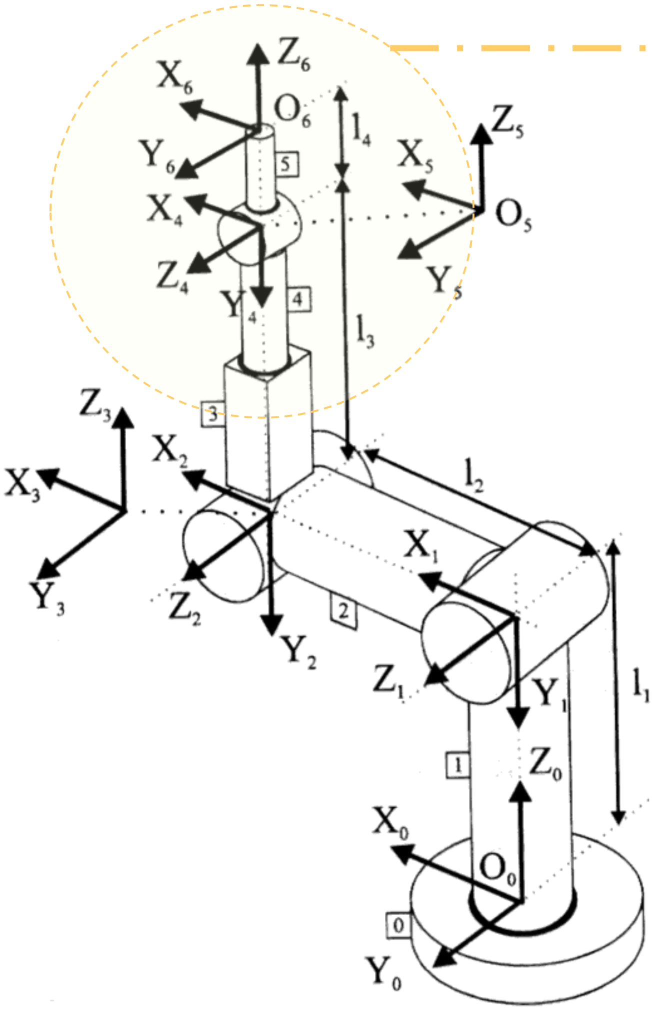
Outline

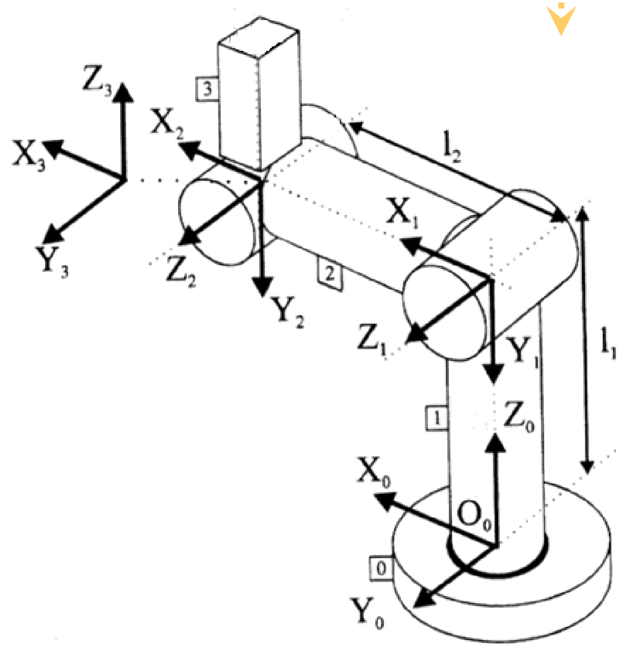
- Inverse Kinematics
- Trigonometric Solutions
- Algebraic Solutions
- **Kinematic Decoupling**
- Summary

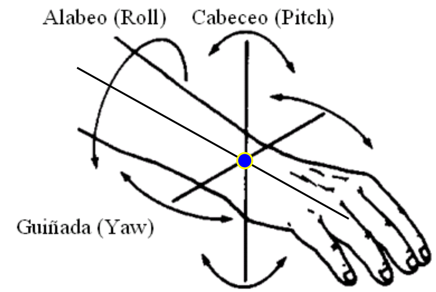
Kinematic Decoupling

- **6-DOF Polar Robot**

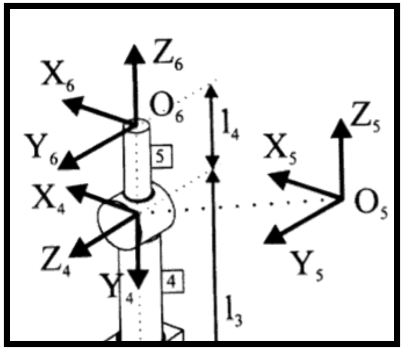
Kinematic decoupling





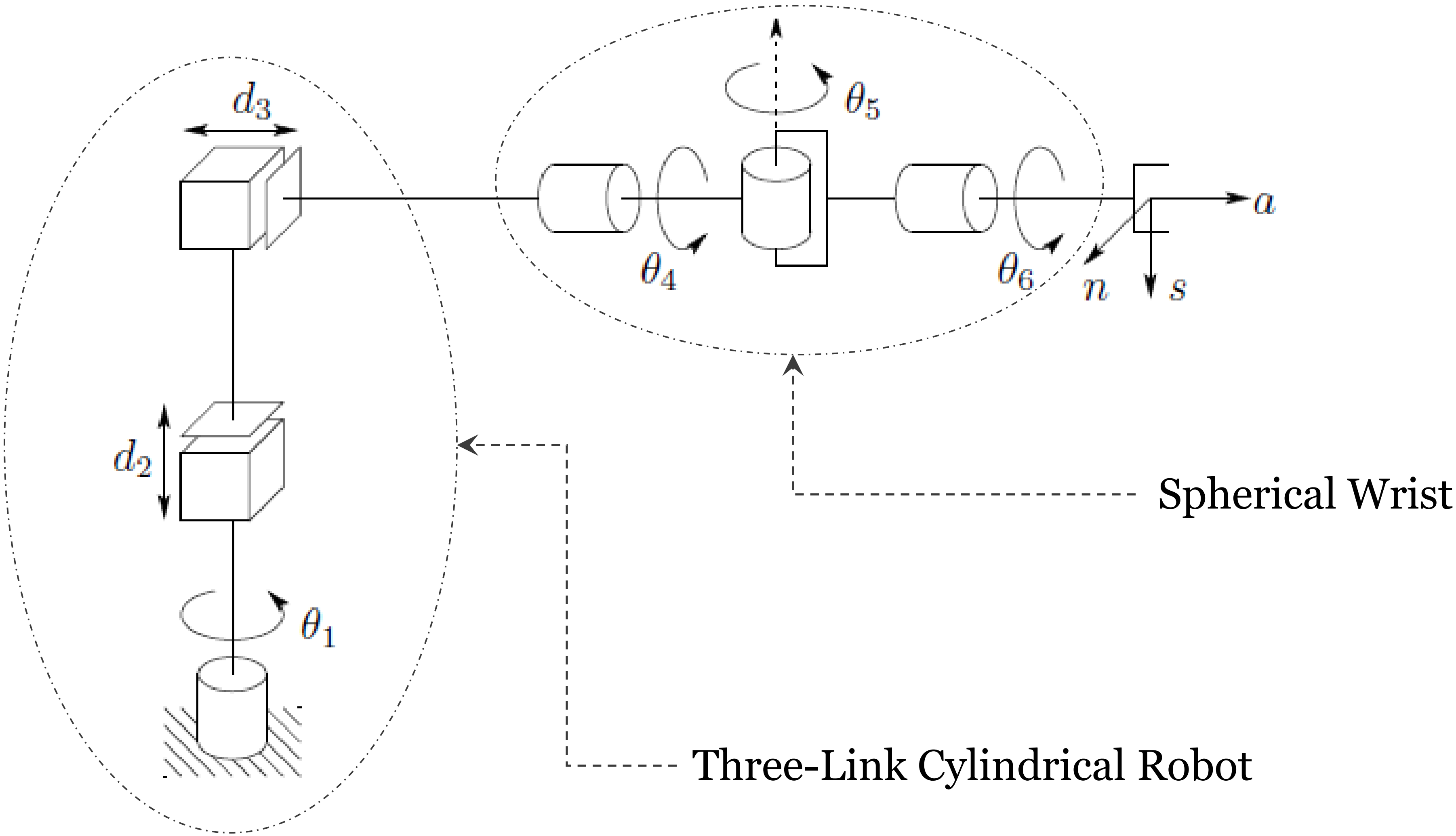


TCP
(Tool Center Point)



Kinematic Decoupling

- **Cylindrical Manipulator with Spherical Wrist**

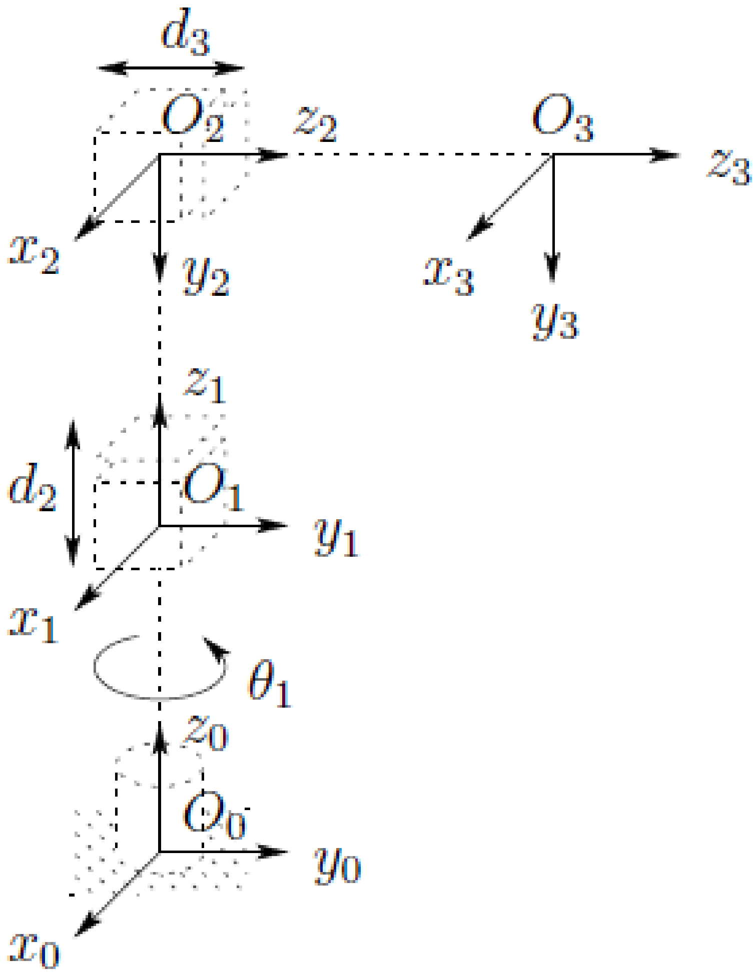


Kinematic Decoupling

- **Three-Link Cylindrical Robot**

Table of D-H Parameters

	Rz	Tz	Tx	Rx
i	θ_i	d_i	a_i	α_i
1	θ_1	d_1	0	0
2	0	d_2	0	-90º
3	0	d_3	0	0



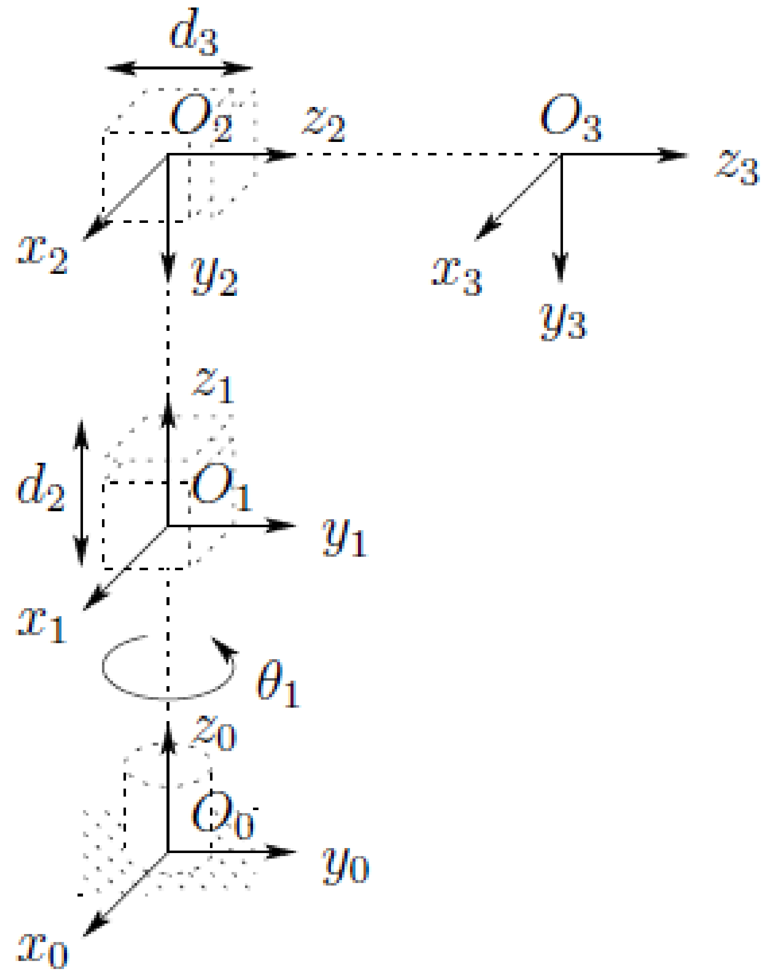
Kinematic Decoupling

- **Three-Link Cylindrical Robot**

$$^{0}A_1 = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{1}A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

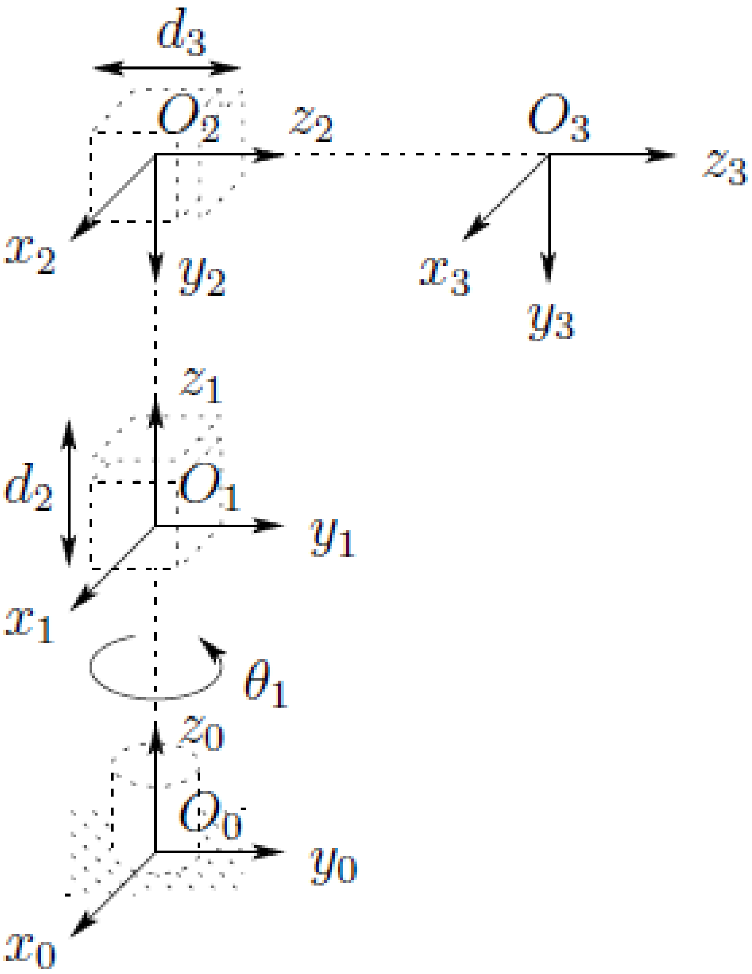
$$^{2}A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Kinematic Decoupling

- **Three-Link Cylindrical Robot**

$$
{}^{0}A_3 = {}^{0}A_1\,{}^{1}A_2\,{}^{2}A_3 = \begin{bmatrix} C_1 & 0 & -S_1 & S_1 d_3 \\ S_1 & 0 & C_1 & C_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



Kinematic Decoupling

- **Spherical Wrist**

  The joint axes z_3, z_4, z_5 intersect at P_w.

$$^{3}A_{4} = \begin{bmatrix} C_4 & 0 & -S_4 & 0 \\ S_4 & 0 & C_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{4}A_{5} = \begin{bmatrix} C_5 & 0 & S_5 & 0 \\ S_5 & 0 & -C_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{5}A_{6} = \begin{bmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

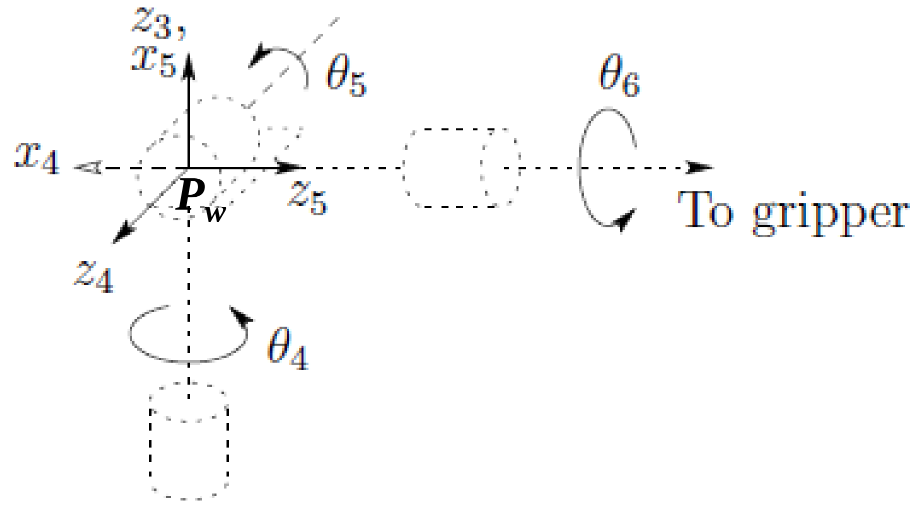
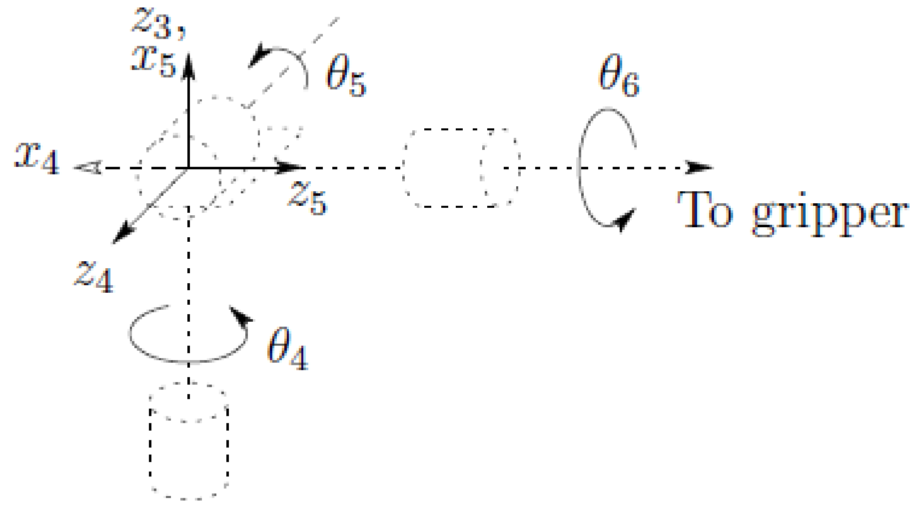


Table of D-H Parameters

	Rz	Tz	Tx	Rx
i	θ_i	d_i	a_i	α_i
4	θ_4	0	0	-90°
5	θ_5	0	0	90°
6	θ_6	d_6	0	0

Kinematic Decoupling

- **Spherical Wrist**

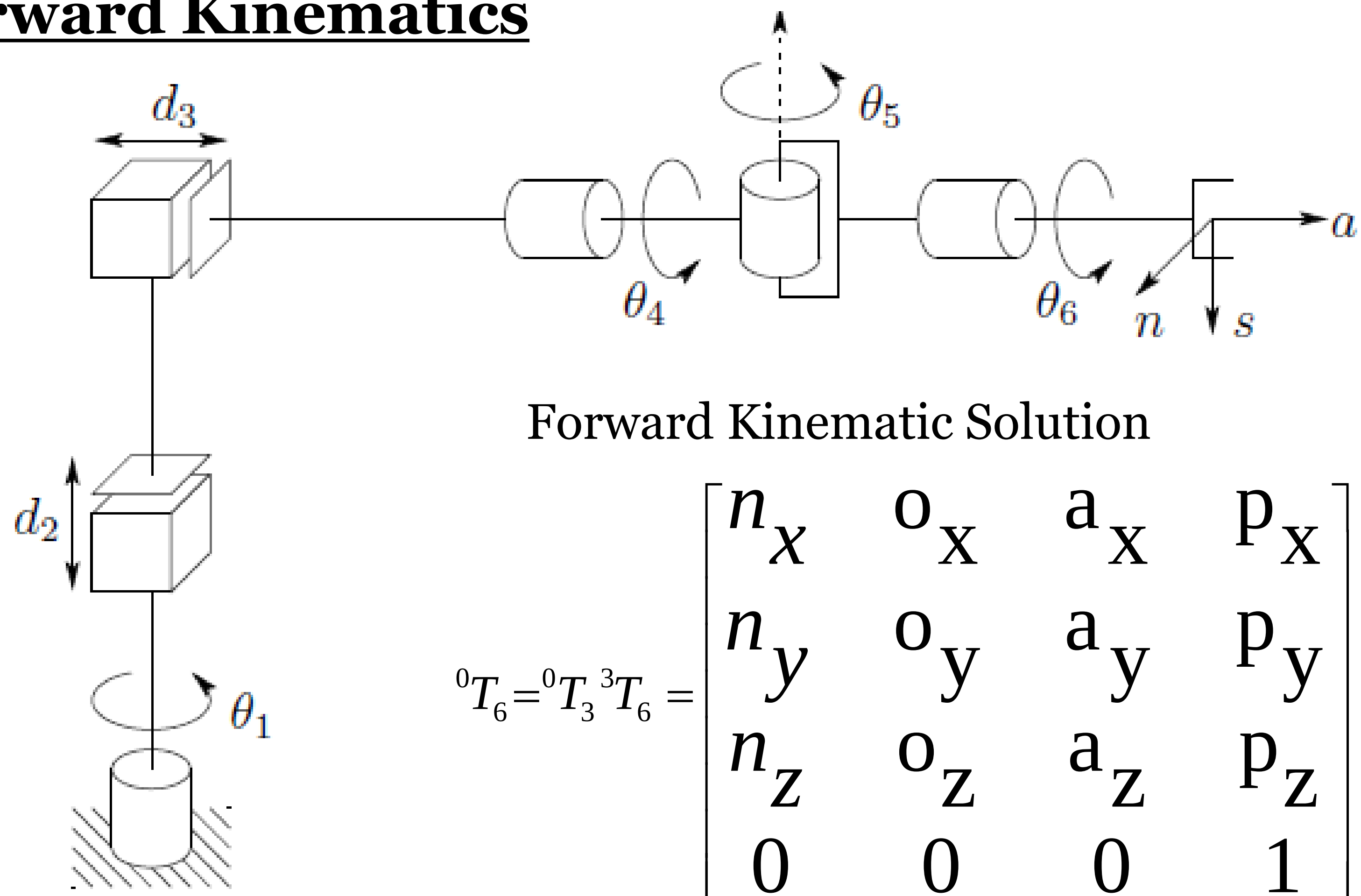


$$
{}^{3}T_{6} = {}^{3}A_{4}\,{}^{4}A_{5}\,{}^{5}A_{6} = \begin{bmatrix} C_4C_5C_6 - S_4S_6 & -C_4C_5C_6 - S_4C_6 & C_4S_5 & C_4S_5d_6 \\ S_4C_5C_6 + C_4S_6 & -S_4C_5S_6 + C_4C_6 & S_4S_5 & S_4S_5d_6 \\ -S_5C_6 & S_5S_6 & C_5 & C_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Kinematic Decoupling

- **Cylindrical Manipulator with Spherical Wrist**

Forward Kinematics



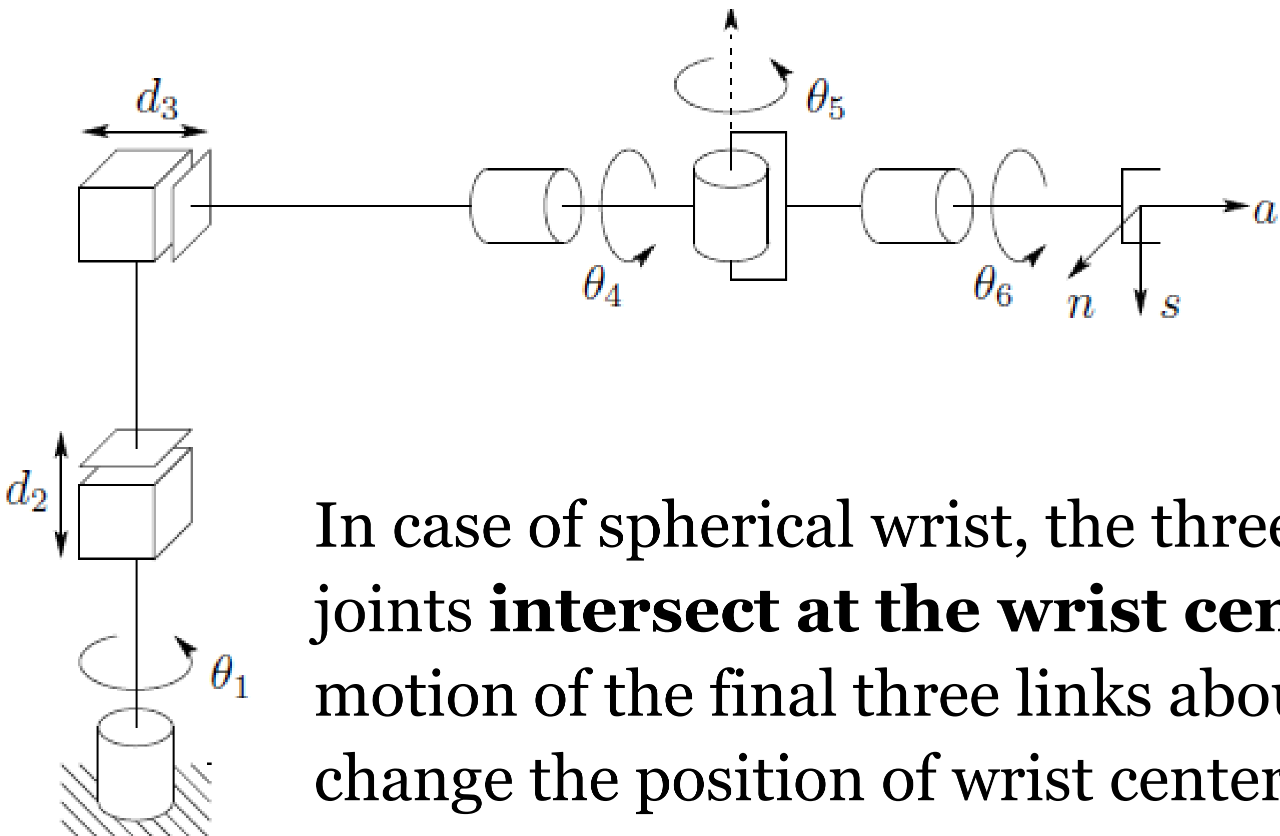
Forward Kinematic Solution

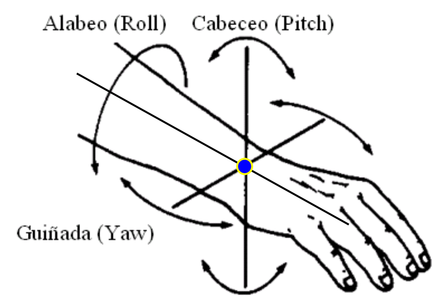
$$ {}^{0}T_{6} = {}^{0}T_{3}\,{}^{3}T_{6} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Kinematic Decoupling

- **Cylindrical Manipulator with Spherical Wrist**

Inverse Kinematics





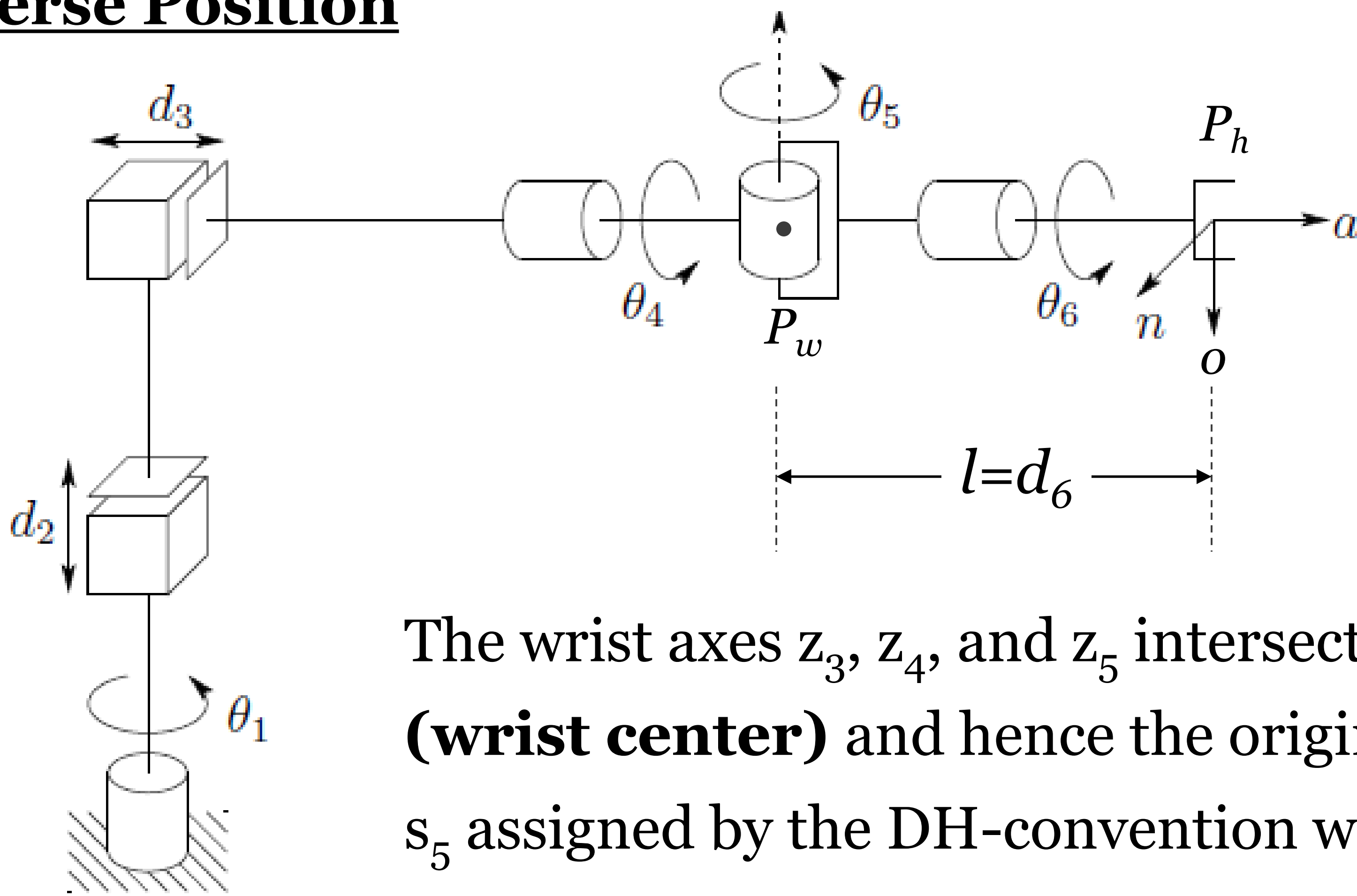
TCP (Tool Center Point)

In case of spherical wrist, the three axes of the wrist joints **intersect at the wrist center** and hence the motion of the final three links about these axes will not change the position of wrist center, and thus, the **position of the wrist center** is a function of only the **first three joint variables**.

Kinematic Decoupling

- **Cylindrical Manipulator with Spherical Wrist**

  Inverse Position

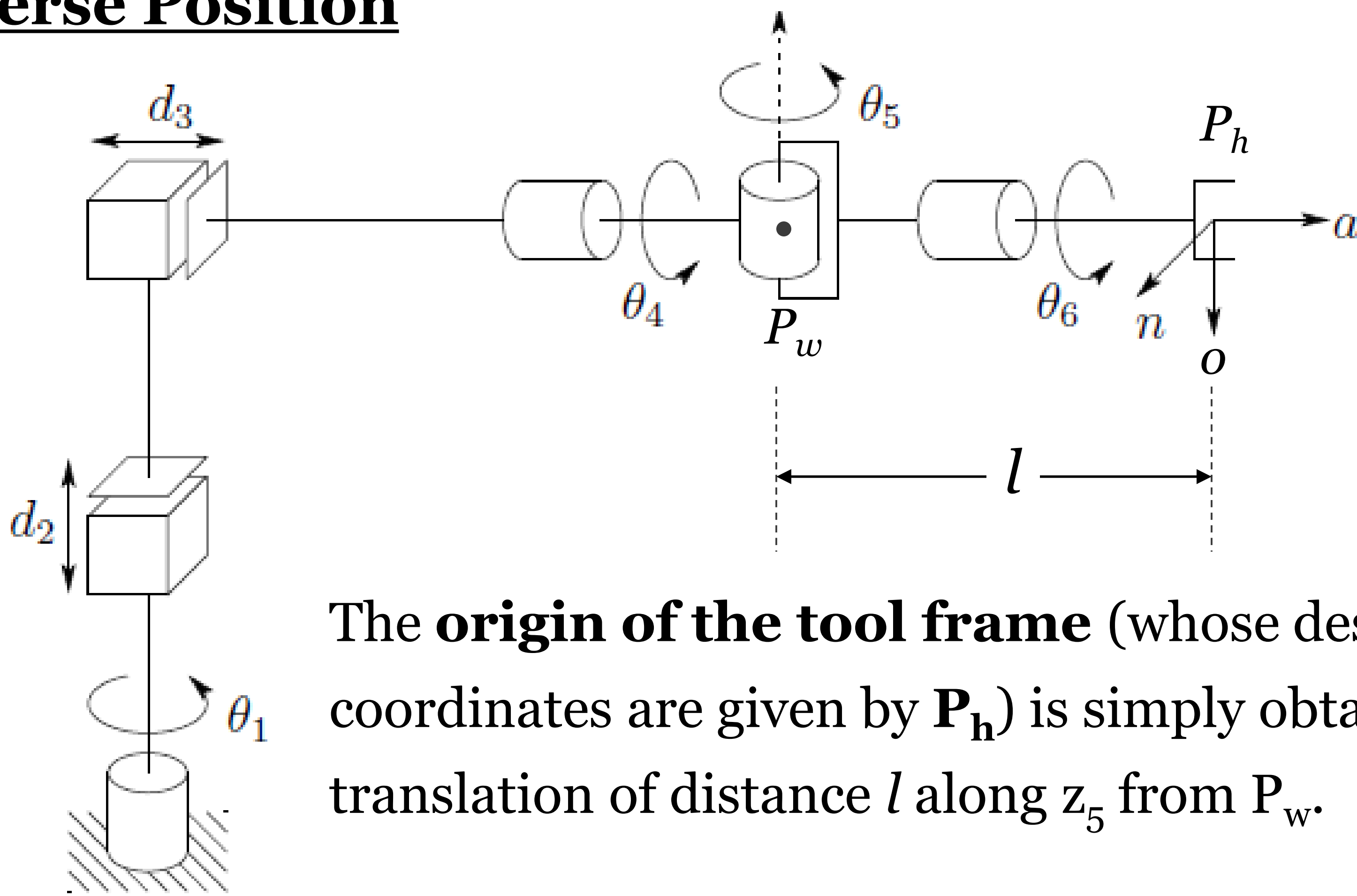


The wrist axes z_3, z_4, and z_5 intersect at $\mathbf{P_w}$ **(wrist center)** and hence the origins s_4 and s_5 assigned by the DH-convention will always be at the wrist center P_w.

Kinematic Decoupling

- **Cylindrical Manipulator with Spherical Wrist**

Inverse Position

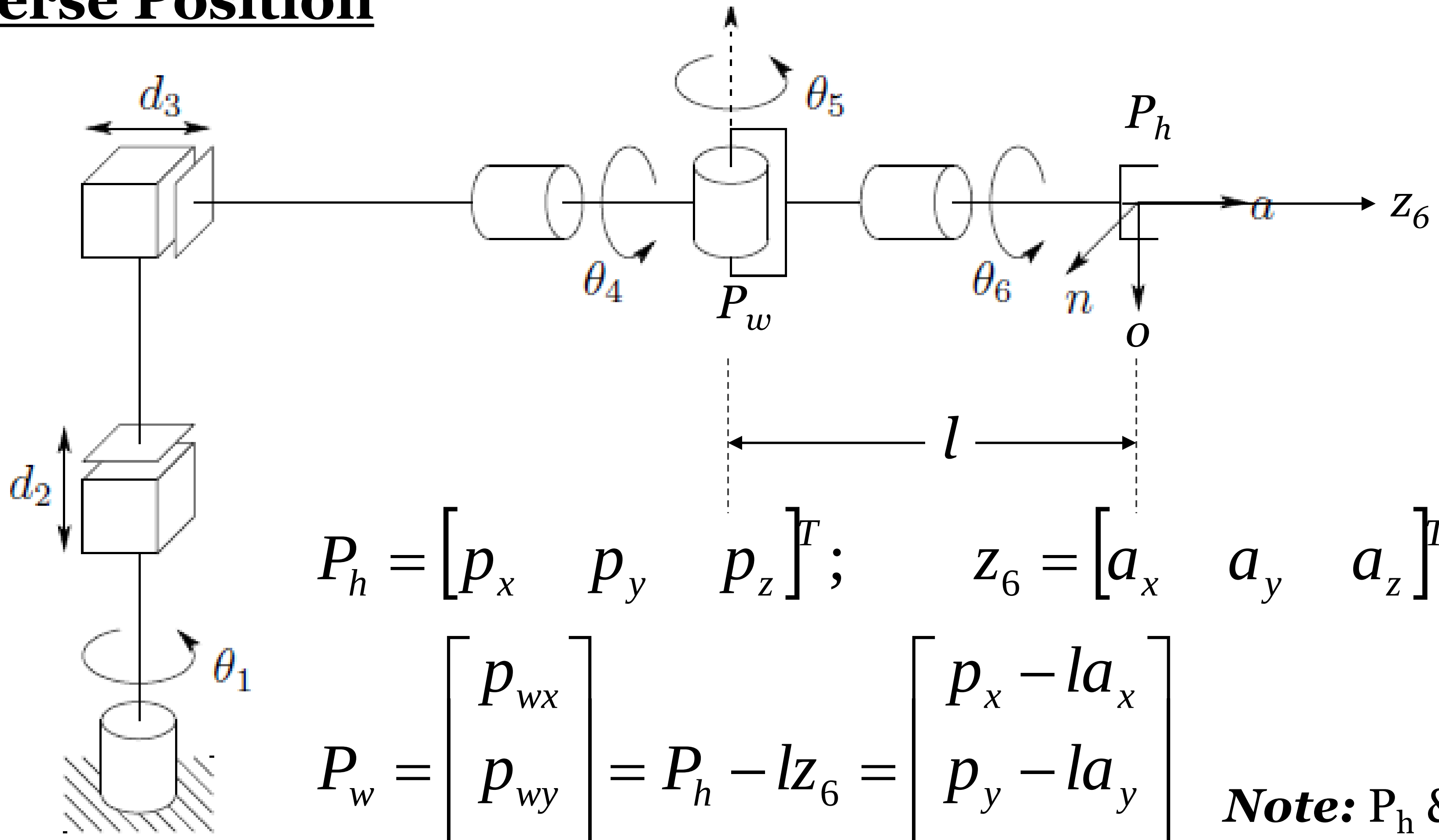


The **origin of the tool frame** (whose desired coordinates are given by $\mathbf{P_h}$) is simply obtained by a translation of distance *l* along z_5 from P_w.

Kinematic Decoupling

- **Cylindrical Manipulator with Spherical Wrist**

Inverse Position



$$P_h = \begin{bmatrix} p_x & p_y & p_z \end{bmatrix}^T; \qquad z_6 = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}^T$$

$$P_w = \begin{bmatrix} p_{wx} \\ p_{wy} \\ p_{wz} \end{bmatrix} = P_h - l z_6 = \begin{bmatrix} p_x - l a_x \\ p_y - l a_y \\ p_z - l a_z \end{bmatrix}$$

Note: P_h & z_6 are given

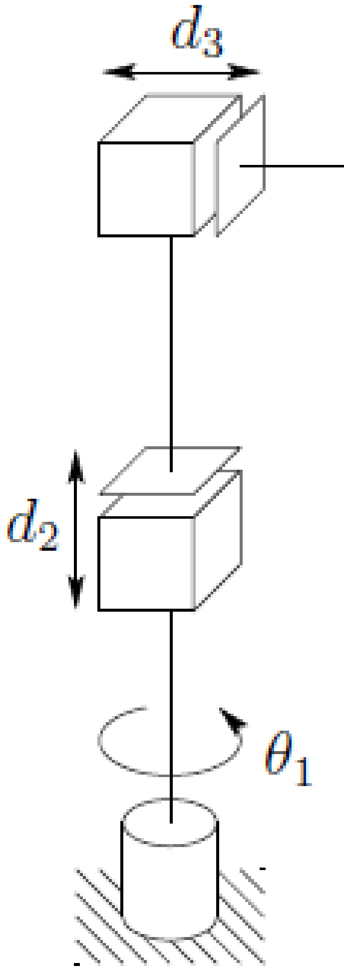
Kinematic Decoupling

- **Cylindrical Manipulator with Spherical Wrist**

Inverse Orientation

$$P_h = \begin{bmatrix} p_x & p_y & p_z \end{bmatrix}^T; \qquad z_6 = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}^T$$

$$P_w = \begin{bmatrix} p_{wx} \\ p_{wy} \\ p_{wz} \end{bmatrix} = P_h - l z_6 = \begin{bmatrix} p_x - l a_x \\ p_y - l a_y \\ p_z - l a_z \end{bmatrix}$$

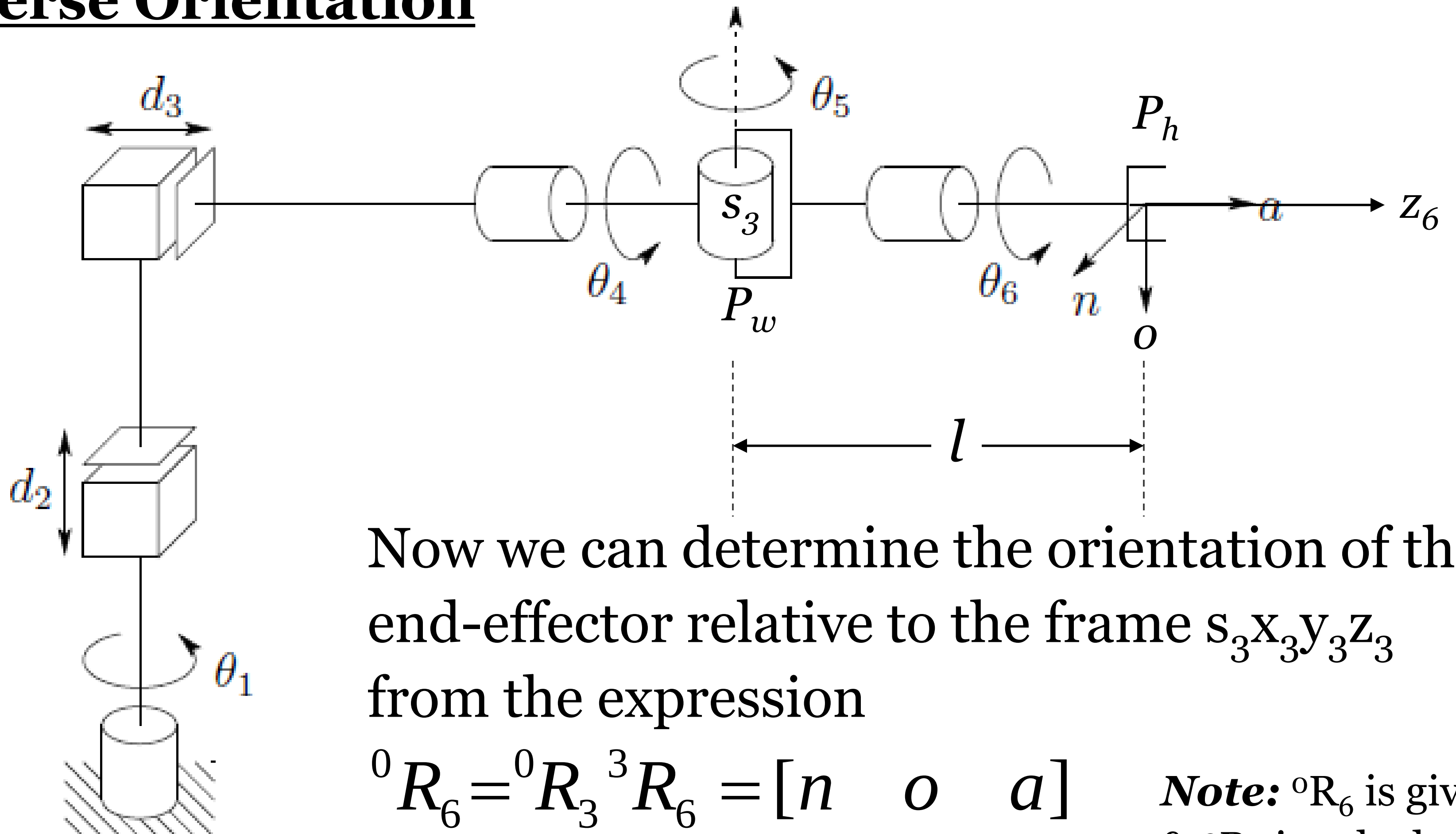


Using the equation we may find the values of the first three joint variables. This determines the orientation transformation **$^0R_3$** which depends only on these first three joint variables θ_1, **d_2 and d_3**.

Kinematic Decoupling

- **Cylindrical Manipulator with Spherical Wrist**

Inverse Orientation



Now we can determine the orientation of the end-effector relative to the frame $s_3x_3y_3z_3$ from the expression

$$^{0}R_6 = {}^{0}R_3\,{}^{3}R_6 = [n \quad o \quad a]$$

$$^{0}R_3^{-1}[n \quad o \quad a] = {}^{3}R_6$$

Note: $^{o}R_6$ is given & $^{o}R_3$ is calculated from 3-DOF arm

Kinematic Decoupling

- **Cylindrical Manipulator with Spherical Wrist**

Inverse Kinematics: Summary

Step 1: Find θ_1, d_2, d_3 such that the wrist center P_w has coordinates given by

$$P_w = \begin{bmatrix} p_{wx} \\ p_{wy} \\ p_{wz} \end{bmatrix} = P_h - l z_6 = \begin{bmatrix} p_x - l a_x \\ p_y - l a_y \\ p_z - l a_z \end{bmatrix}$$

Step 2: Using the joint variables determined in Step 1, evaluate ${}^0R_3$

Inverse Position Kinematics

Step 3: Find a set of Euler angles corresponding to the rotation matrix

$${}^0R_3^{-1}[n \quad o \quad a] = {}^3R_6$$

Step 4: Use ${}^3R_6$ to find θ_4, θ_5, θ_6 .

Inverse Orientation Kinematics

Outline

- Inverse Kinematics
- Trigonometric Solutions
- Algebraic Solutions
- Kinematic Decoupling
- **Summary**

Summary

- While forward kinematics determines the end-effector position and orientation in terms of the joint variables, inverse kinematics is concerned with finding the joint variables in terms of the end-effector position and orientation.
- In general, inverse kinematics problem is more difficult than the forward kinematics problem.
- Although the general problem of inverse kinematics is quite difficult, it turns out that for manipulators having six joints, with the last three joints intersecting at a point (such as the Stanford Manipulator above), it is possible to decouple the inverse kinematics problem into two simpler problems, known respectively, as inverse position kinematics, and inverse orientation kinematics.